Modelling Pendulums and Analyzing the Effects of Amplitude, String Length, and Mass on the Period

INTRODUCTION

In classical mechanics, a pendulum refers to a mass hanging and allowed to swing freely about a pivot. Due to the restoring force of gravity acting on the mass when displaced from its equilibrium position, pendulums are harmonic oscillators, exhibiting periodic motion. The motion of pendulums can be modelled by the equation for damped harmonic motion: $\theta(t) = \theta_0 e^{-t/\tau} \cos\left(2\pi \frac{t}{T} + \varphi_0\right)$. This model predicts four key features: (1) exponential decay of amplitude, (2) independence of period from amplitude, (3) dependence of period on string length, and (4) independence of period from mass. This report tests the validity of the mathematical model by analyzing these features and how well they fit to data taken from a homemade pendulum.

The decay of the amplitude was studied and can be represented by the quality (Q) factor of the pendulum. Q was determined in two ways: (1) by the equation $Q = \pi \frac{\tau}{T}$, where the parameters τ (time constant of decay) and T (period) can be found by finding the equation of the curve of best fit to the collected data, or (2) by counting the number of oscillations until the amplitude of the pendulum's motion reduces to $e^{-\frac{\pi}{2}}$ of the release angle, then multiplying that value by 2. These two methods produced Q factors of 170 ± 20 and 160 ± 10, respectively.

For larger amplitudes (θ_0), there is dependence of period on amplitude which can be modelled by a power series, $T = T_0 + B\theta_0 + C\theta_0^2$, but for small enough amplitudes, a pendulum's period can be given by the equation: $T = 2\pi\sqrt{\frac{L}{g}}$, where L is the pendulum length and g is gravitational acceleration[1]. Data taken on the period with respect to amplitude was fitted to the power series and a range of θ_0 was identified where B and C could be considered negligible (i.e. where the latter equation holds true). T_0, B, and C were found to be 1.369 s ± 0.007 s, 0.08 s/rad ± 0.05 s/rad, and 0.059 s/rad^2 ± 0.006 s/rad^2, respectively. It was determined that B and C could be considered negligible when $|\theta_0| \leq 0.5\, rad$. The second equation also reveals a relationship between period and string length that can be approximated by $T = 2\sqrt{L}$, or equivalently, a power law equation: $T = k(L_0 + L)^n$, where k = 2, L_0 = 0, and n = 0.5. Fitting the collected data to the power law equation, k, L_0, and n were found to be 1.83 s/m$^{0.3}$ ± 0.07 s/m$^{0.3}$, -0.06 m ± 0.01 m, and 0.30 ± 0.04, respectively. These values do not agree with the theoretical values and therefore the pendulum cannot be modelled by the predicted power law equation. The absence of a mass variable in the period equation reflects independence of period from mass. Collected data were consistent with this prediction; no clear trend/pattern was found between the period and the mass.

METHOD

To create the pendulum, a mechanical pencil was taped down to a desk with around 1.5 cm hanging off the edge to act as the pivot. Heavy binders were placed on top of the pencil to ensure minimal movement. A loop was made on one end of a piece of thread to hang onto the pencil. The other end was tied to a paper clip shaped into a hook to easily hook on or remove masses. A locker lock was used as the mass. A piece of paper was taped to the desk behind the pendulum to mark angles (using a protractor). Figure 1 presents a visual of this setup.

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[1] Pendulum. (2020, November 14). Retrieved November 18, 2020, from https://en.wikipedia.org/wiki/Pendulum

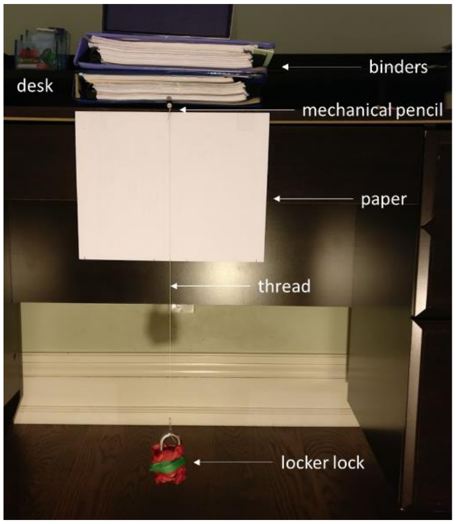

Figure 1. Pendulum Setup. The pendulum swings parallel to the pane of the photo. Angles were measured positive in the counterclockwise direction relative to the equilibrium position. Red and green tissue paper was wrapped onto the lock to provide unique pixels for the tracking software to track.

Throughout this experiment, my phone's camera was used to film the pendulum for data collection. A Python program was used to fit data to equations and calculate their parameters.

To determine Q, the pendulum was filmed being released from an angle of $\frac{\pi}{6}$ rad [2] and was left to swing until its amplitude reduced to $e^{-\frac{\pi}{2}}$ of the release angle ($\sim 6°$). The number of oscillations it took was counted by reviewing the video and multiplied by 2. Analyzing the decay of amplitude with time also required collecting data on the motion of the lock. A physics video analysis software called *Tracker* was used to track the angular position of the lock (angle made with its equilibrium position) frame by frame and the elapsed time since initial release. A curve of best fit was determined with this data and the parameters of the curve were then used to determine Q.

To examine the relationship between the period and amplitude, period and string length, and period and mass, the pendulum's period was timed for multiple trials with varying values of the independent variable being studied. For all trials, the pendulum was timed for one oscillation to determine its period due to the determined value of the Q factor – the pendulum's amplitude reduces by around 2% after one oscillation (the equation for this calculation is discussed in the analysis and is also derived in Appendix A.2), which is less than the instrumental uncertainty of the protractor used (discussed in Analysis). For testing string length and mass, the pendulum was released from an angle of $\frac{\pi}{12}$ rad each test to ensure that the amplitude had negligible effect on the period based on the value of C obtained from testing the amplitude (shown in Appendix A.3).

For testing the effect of amplitude, values of θ_0 from $\frac{\pi}{12}$ rad to $\frac{\pi}{2}$ rad were tested, with θ_0 increasing by an increment of $\frac{\pi}{12}$ rad each time. With each positive θ_0, the corresponding negative θ_0 was also tested to check for asymmetry within the pendulum. Collected data on the period was fitted to a power series and the values for T_0, B, and C were determined.

Starting from a thread length of 0.10 m, seven different thread lengths were tested to study the effect of string length on period, increasing length by 0.05 m each time. Data was to the power law function, providing values for k, L_0, and n.

For testing the effect of mass, quarters (Canadian coin) were used as the mass instead of the locker lock. One quarter has a mass of 4.40 g[3]. To hang them from the thread, a piece of a

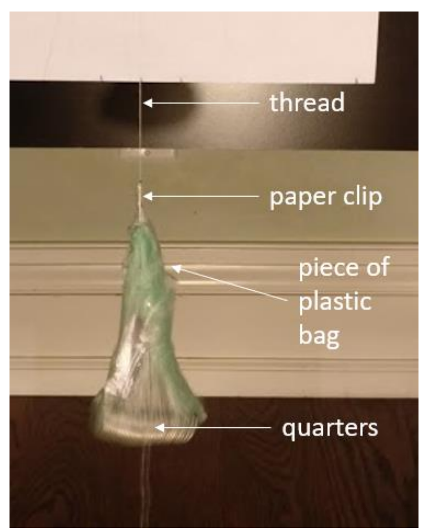

Figure 2. Modified mass setup to allow for changeable mass.

[2] The release angle was decided arbitrarily but within reason – significantly larger angles would have taken too long to record data for, significantly smaller angles would not have provided enough data.

[3] Quarter (Canadian coin). (2020, September 27). Retrieved November 18, 2020, from https://en.wikipedia.org/wiki/Quarter_(Canadian_coin)

plastic bag was used to create a small bag-like contraption that could carry the coins and hook onto the paperclip. The resulting pendulum length of this setup was 0.4290 m ± 0.0005 m. A visual of this modified setup is presented in Figure 2. Five different masses were tested: starting at 8.80 g (2 quarters), the mass was doubled each trial; the heaviest mass was 140.8 g (32 quarters). Collected data was compared with the theoretical period and analyzed for any significant trends/patterns.

RESULTS

The data collected on the angular position of the pendulum was fit to the given mathematical model of the pendulum, as seen in Figure 3, to determine the parameters τ and T. Using these values, the resulting Q factor was calculated to be 170 ± 20 (Appendix A.1). Using the method of counting the number of oscillations, the Q factor was determined to be 160 ± 10.

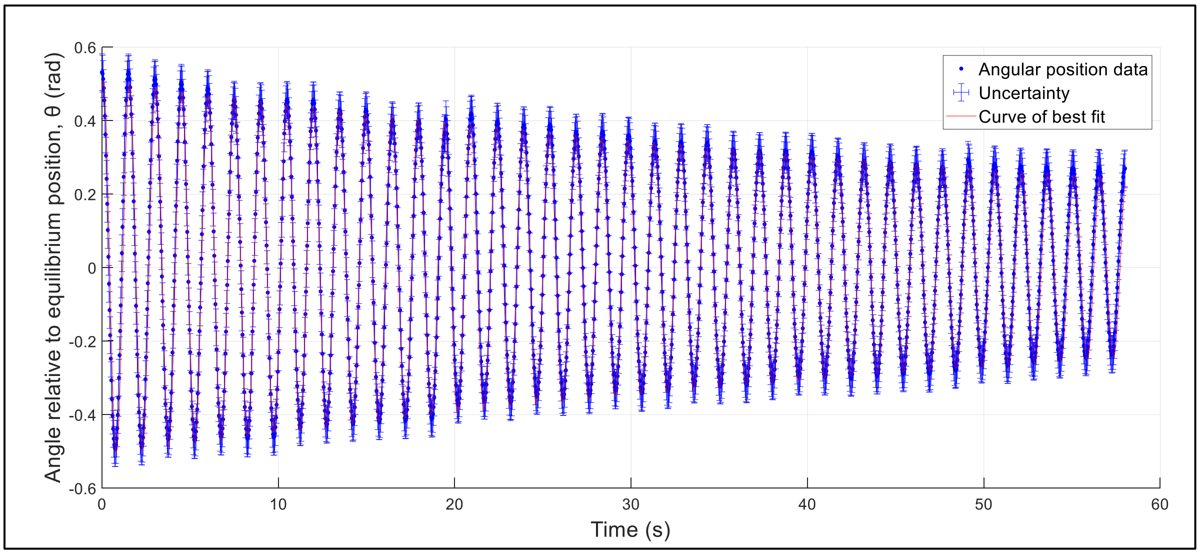

Figure 3. The curve of best fit modelling the pendulum's angular position with respect to time. The angular position over time takes the form of an exponentially decaying sinusoidal curve, as predicted, represented by the curve of best fit. The curve is represented by the equation: $\theta(t) = -0.508e^{-\frac{t}{78.4}}\cos\left(2\pi\frac{t}{1.49} + 2.85\right)$.

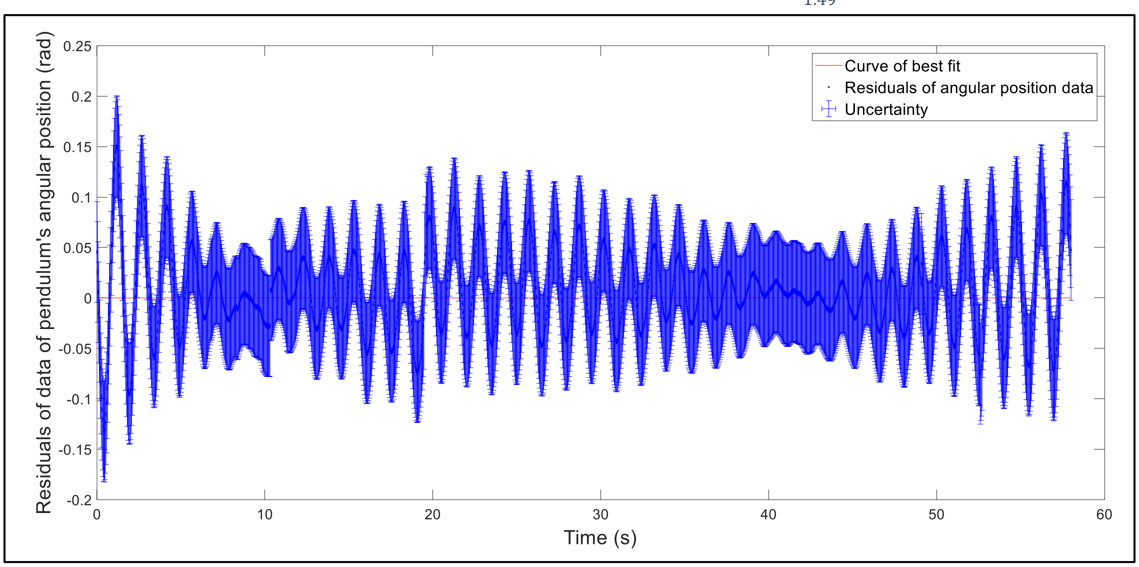

Figure 4. The residuals of the pendulum's modelled angular position with time are shown to produce a sinusoidal pattern. The error values reduce at around t = 9 s and t = 42 s. This pattern is caused by a slight phase shift between the best fit curve and the data shown in the following figures.

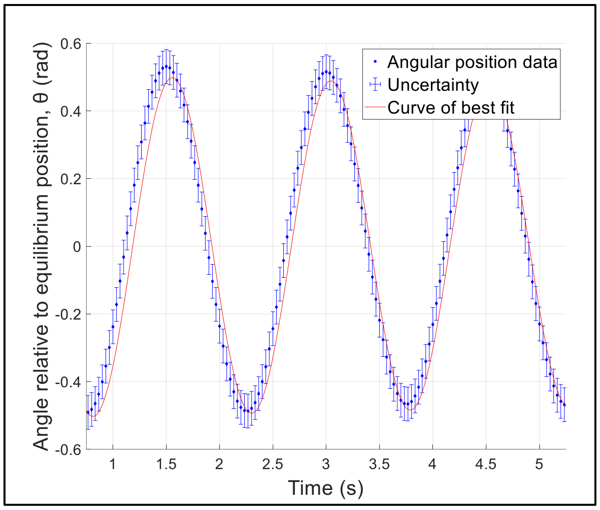

Figure 5. The curve of best fit modelling the pendulum's angular position with time from t = 1 s to t = 5 s. Data points are shifted to the left of the curve of best fit, causing a sinusoidal pattern in the residuals. The cause of the shift is not known for certain but could be attributed to flaws in the tracking software used.

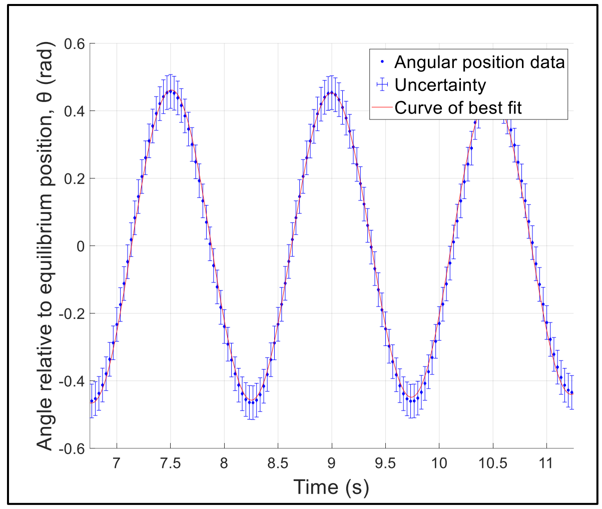

Figure 6. The curve of best fit modelling he pendulum's angular position with time from t = 7 s to t = 11 s. At around t = 9 s, the data points seem to shift into phase with the curve of best fit briefly, causing a reduction in error values, before shifting to the right of the curve. At around t = 42 s, the data points shift back to the left of the curve.

In the process of gathering data for examining the relationship between the pendulum's period and amplitude, the thread connecting the lock to the pivot snapped. A new pendulum was created and an effort was made to replicate the thread length as precisely as possible, but there is still a noticeable discrepancy in the results. The Q factor of the new pendulum was measured to be the same as the old one. The following data was collected with the new pendulum.

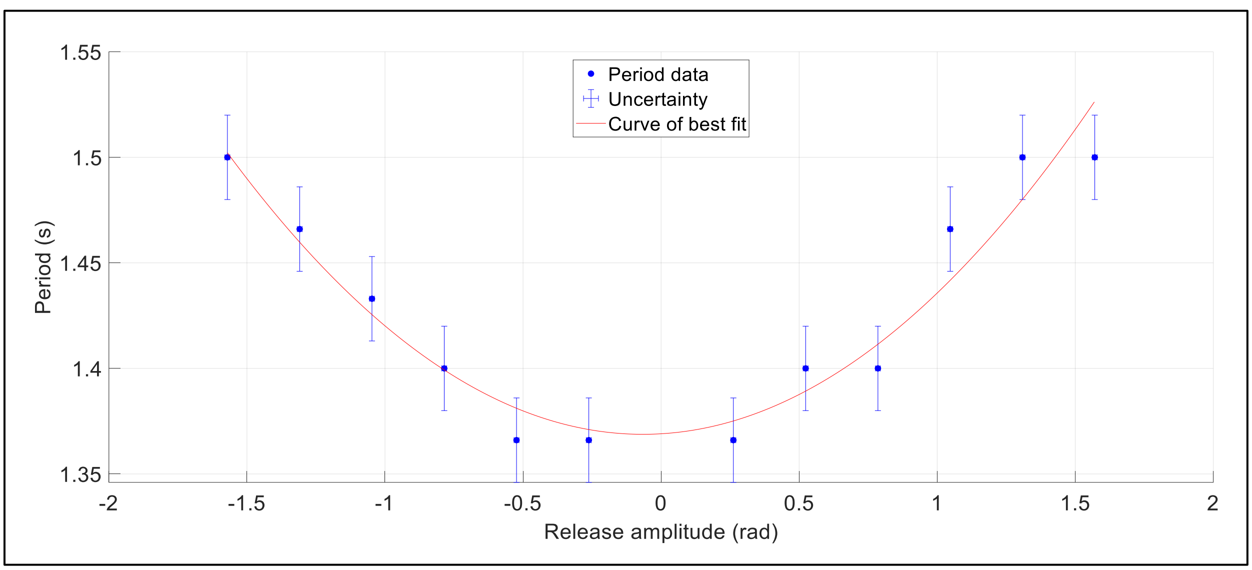

Figure 7. The period of the pendulum follows a parabolic pattern in relation to the release amplitude. The curve of best fit is represented with the equation: $T(\theta_0) = T_0 + B\theta_0 + C{\theta_0}^2$, where $T_0 \cong 1.369\, s \pm 0.007\, s$, $B \cong 0.08 \frac{s}{rad} \pm 0.05 \frac{s}{rad}$, and $C \cong 0.059 \frac{s}{rad^2} \pm 0.006 \frac{s}{rad^2}$.

As seen in Figure 7, data of the pendulum's period as a function of release amplitude seems to form a parabolic shape, opening upwards, centered roughly around 0 rad. Fitting the data to a power series, the value of T_0, B, and C were found to be 1.369 s ± 0.007 s, 0.08 s/rad ± 0.05 s/rad, and 0.059 s/rad^2 ± 0.006 s/rad^2, respectively.

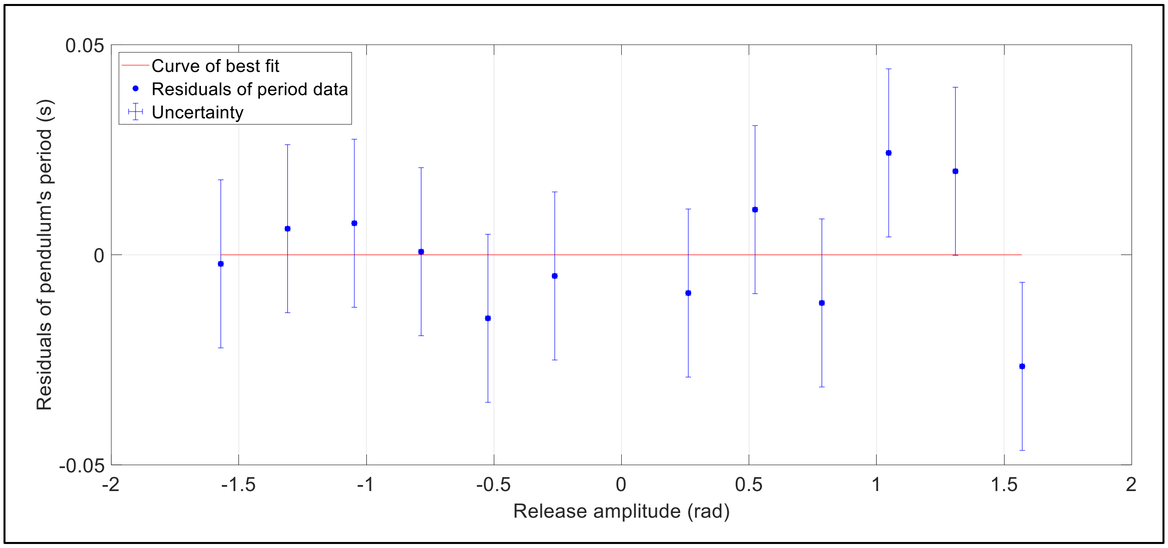

Figure 8. The residuals of the data collected on the pendulum's period when released from varying initial amplitudes shows no prominent patterns or features, indicating that there were no unusual causes for errors in the data – the curve of best fit is well fit to the data.

A clear dependence of period on string length was observed when testing the pendulum's length. While it can be seen in Figures 9-11 that the pendulum fits a power law equation very well (the fitted curve lies within the uncertainty for 6 out of 7 data points), the values of the parameters do not agree with the theoretical values; the values of the parameters k, L_0, and n were found to be 1.83 s/m$^{0.3}$ ± 0.07 s/m$^{0.3}$, -0.06 m ± 0.01 m, and 0.30 ± 0.04, respectively. Therefore the predicted power law equation is inconsistent with my results.

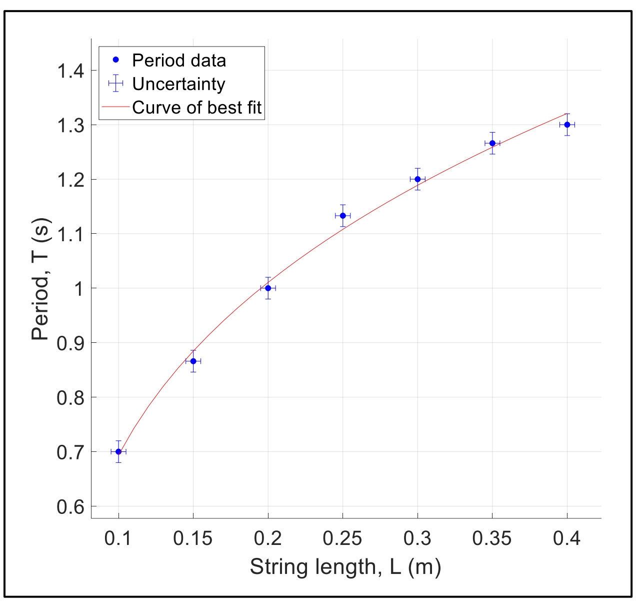

Figure 9. The relationship between the pendulum's period and string length follows a curve represented by the following power law equation: $T = k(L_0 + L)^n$, where k = 1.83 s/m0.3 ± 0.07 s/m0.3, L0 = -0.06 m ± 0.01 m, and n = 0.30 ± 0.04.

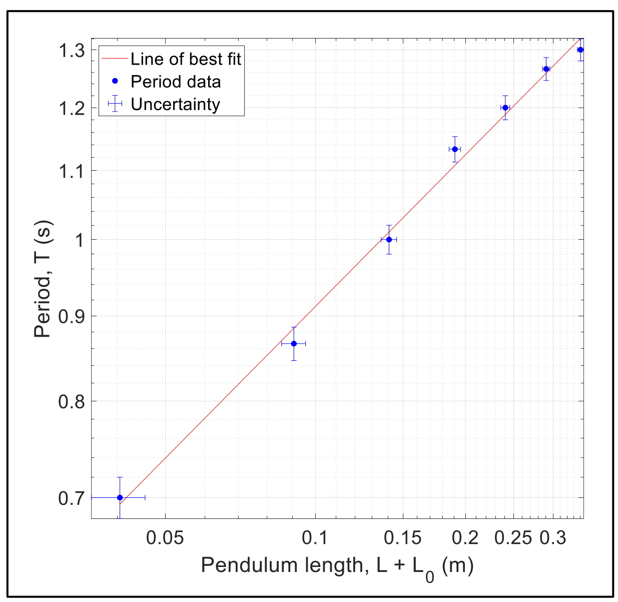

Figure 10. The relationship between the period and effective string length (pendulum length) is plotted on logarithmic axes.

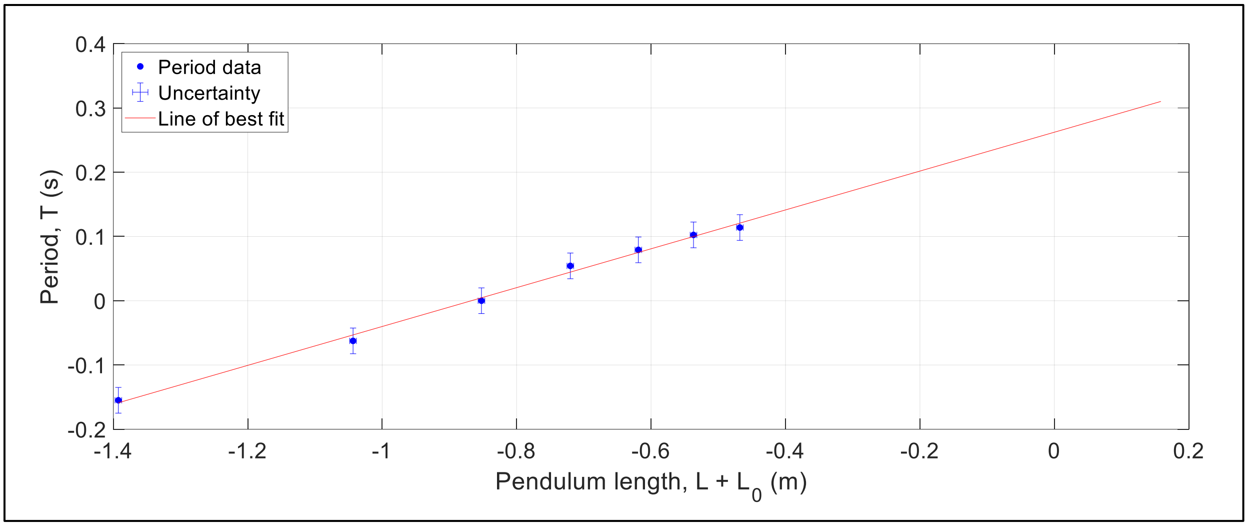

Figure 11. A similar plot to Figure 10, except instead of plotting on logarithmic axes, the logarithms of the data points are taken first, then plotted on a linear axis. Here, it is easier to see how the parameters k and n are represented in a log-log plot; n = 0.30 is the slope, log(k) = 0.26 is the y-intercept.

When studying the relationship between period and mass, the equation for a pendulum's period ($T = 2\pi\sqrt{\frac{L}{g}}$) was first used to calculate a predicted value for the period of 1.31 s ± 0.04 s, given that the pendulum length of the setup was 0.4290 m ± 0.0005 m. Plotting the collected data and the theoretical value of the period on the same graph (Figure 12), it can be seen that the data points are consistent with the theoretical value, as the uncertainties of the data points and the theoretical period overlap. Furthermore, four of the five data points were recorded to be the exact same value.

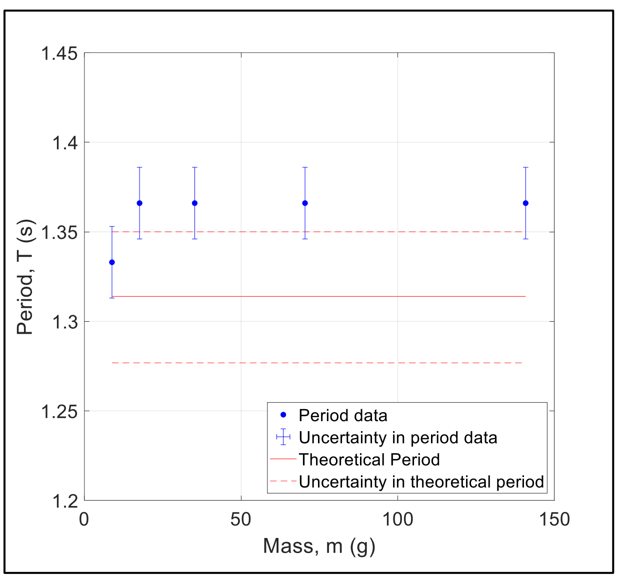

Figure 12. The data collected on the period in relation to the mass is plotted. There is no clear relationship between the period and mass of the pendulum.

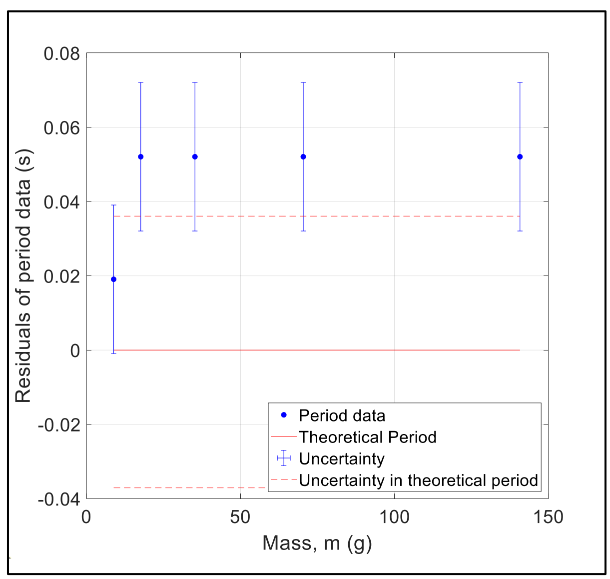

Figure 13. The residuals of the period data in relation to the mass show no clear trend or pattern. This implies that the period of the pendulum is independent of the mass.

ANALYSIS

The two values of Q found (170 ± 20 and 160 ± 10) agree very well with each other as they both lie within each other's uncertainties – both are acceptable values with respect to each other. These values had significant impact on the way by which measurements in the succeeding experiments were taken since it defines how long the pendulum can accurately represent simple harmonic motion (without decaying amplitude) for. More specifically, the number of oscillations that occur before the amplitude of the swing reduces below some given constraint can be represented as a function of Q (Appendix A.2). For example, given that 95% of the initial amplitude is still considered "accurate", measurements can be taken for around 2.5 cycles before they will be considered unacceptable. This makes Q valuable for determining experimental methods to ensure a given degree of accuracy.

While the overall shape followed an expected exponentially decaying sinusoidal pattern, an interesting and unexpected result in the angular position data appeared in the residuals of the curve of best fit – the residuals displayed a sinusoidal pattern when the expected result was random noise. By inspecting the graph of the best fit curve and the data points, it is clear that the cause of this pattern is a phase shift between the fit and the data, but the cause of the shift is not known for certain. It is also interesting to note that approaching t = 9 s, the data begins to shift in phase with the fit, reducing the error values, and after t = 9 s, the data continues to shift to the right, once again increasing the error values. This occurs once more at t = 42 s, where the data shifts back to the left of the fit.

A possible cause for these shifts could be in the tracking software. I noticed, while running *Tracker*, the point on the lock used to measure the angle was not always at the centre of mass of the lock. This is because the software uses unique pixels to track objects, and due to my phone's camera quality, the lock's image was displayed by many pixels, none of which were significantly different from one another. A shift in the data to the right of the fit could imply that the software's detected centre of mass shifted to the left, and vice versa. If this is the cause of the shift in the data, it reinforces the accuracy of the mathematical model since the sinusoidal pattern in the residual graph would not have appeared if the software did not incorrectly track the lock's centre of mass.

Data on the relationship between the period and amplitude fit nicely to a parabolic curve – the curve lay within the uncertainties of a large majority of the data points. Regarding the implications of the values of the parameters of the fitted curve: B is shown to be consistent with 0 since its value is less than two times its uncertainty. Therefore, the pendulum is symmetric. Conversely, the value of C is nearly ten times its uncertainty, indicating a dependence of the pendulum's period on the release amplitude. For the purpose of controlling variables in succeeding experiments, it was useful to find a range of θ_0 small enough where C can be considered insignificant. To have a negligible effect on the period, $C\theta_0^2$ should be less than or equal to the measured time uncertainty in these experiments, which is 0.02s (explained below along with other sources of uncertainty). This way, it will not be the factor to limit the experiment's accuracy regarding measured durations of time. Plugging the value of C into this inequality, it is found that $|\theta_0| \leq 0.582\, rad$ (Appendix A.3). Rounding up to keep one significant digit would give a value of 0.6 rad, but to stay conservative, it is better to round down to ensure that the value of C will have negligible effect on the period: $\therefore |\theta_0| \leq 0.5\, rad$.

The predicted parameters of the power law model for the relationship between the period and string length disagreed with my experimental results. The discrepancy in the value of L_0 may be attributed to the difference between *string length* (length of the string) and *pendulum length* (length from the pendulum's pivot to the center of mass of the mass). The predicted value of $L_0 = 0$ assumes that the measured string length is equal to the pendulum length, i.e. the mass is a point mass. The fact that the collected data fit so well to the curve with a non-zero L_0 value implies that there must have been a consistent error in the measurement of the pendulum length, which is logical considering that the lock used as the mass is *not* a point mass and therefore only measuring the string length of my pendulum would not be equal to the pendulum length. Following this logic, the pendulum length should be obtained by taking the sum of L and L_0. By observing the setup in Figure 1, it can be roughly estimated that the pendulum length should be a few centimeters longer than the string length, but in this case, since the value of L_0 is negative, the pendulum length would be shorter than the string length, which does not make sense. The cause of this fallacy in reasoning is because it assumes that the other two parameters, k and n, agreed with the predicted values of 2 and 0.5, which they did not. Therefore it is difficult to determine what L_0 represents in this context.

While the experimental results did not agree with the predicted model, it is interesting to note the result of fitting the data to the equation $T = 2(L_0 + L)^{0.5}$, where we assume that values of k = 2 and n = 0.5 accurately model the period of the pendulum, and the only parameter to fit is L_0. Fitting the data to this equation using the Python program results in an L_0 value of 0.04 m ± 0.01 m. Using the logic presented in the paragraph above and summing L and L_0 to obtain the pendulum length, it would be consistently around 4 cm longer than the measured string length, which is a reasonable estimate given the size of conventional locker locks and matches our previous estimate of “a few centimeters”.

Finally, studying dependence of period on mass, no clear pattern was found in the data, as well as in the residuals; the residuals essentially form a flat line with the exception of a single data point. This implies that there is no relationship between the period and mass of the pendulum. Therefore, mass having no effect on the period is an acceptable simplification for my pendulum. Regarding the discrepancy between the predicted period and the experimental period, it is likely that the error is a result of minute mistakes in measuring due to the lack of a trend within the data.

Uncertainties

The most common source of uncertainty was instrumental uncertainty. The protractor, used in all experiments, provided a measurement uncertainty of ± 0.009 rad (smallest increment on the protractor is 1°). The phone’s camera, which was also used in all experiments, had a maximum frame rate of 30 fps, meaning the smallest increment of time measurable was 0.033 s, producing a time uncertainty of ± 0.02 s. To reduce time uncertainty, a video recorder with a higher frame rate should be acquired. Additional uncertainties are discussed below:

In both methods of determining Q, as a result of its instrumental uncertainty using the protractor while measuring the final angle produced an 8% uncertainty. Therefore an 8% uncertainty was used when determining the uncertainty of Q for the counting method. A more precise protractor should be used to improve results.

In the method of determining Q involving the tracking software, a source of uncertainty was the inaccuracy of the detected centre of mass. I used the software’s built-in protractor and measured the greatest angle of variance possible as ± 0.05 rad, i.e. if the detected centre of mass was at the very edge of the lock. Producing an uncertainty of around 10%, this was the largest source of uncertainty and was therefore used when determining the uncertainty for Q for this method. The lock should be modified visually to provide the tracking software with a smaller and more unique image to track, located at the centre of the lock’s mass (e.g. placing a unique sticker on the face of the lock). This will prevent inaccuracies in the detected centre of mass, improving results.

Measuring the pendulum's string length and mass was affected by the instrumental uncertainty of the measuring tape used to measure string length. It produced an uncertainty of ± 0.0005 m, while the source providing the information on the mass of the quarter was only accurate to the nearest 0.01 g, producing an uncertainty of ± 0.005 g. When examining the relationship between the period and mass, there was also uncertainty behind the calculation of the theoretical value of the period. In measuring the pendulum length, two sources of uncertainty arose: (1) the instrumental uncertainty of the measuring tape as discussed before, and (2) the uncertainty of where the center of mass of the quarters were. Since the quarters were all aligned horizontally and are 0.02388 m in diameter, I was only confident in my pendulum length measurement to the nearest 0.02388 m. Propagating this uncertainty to calculate the shortest and longest theoretical period possible given a measured pendulum length of 0.429 m, I determined the uncertainty of the theoretical period to be ± 0.04 s.

CONCLUSION

As predicted, the angular position of the pendulum follows an exponentially decaying sinusoidal pattern. A plausible explanation for the unusual sinusoidal pattern seen in the residual graph is the shifting of the software's detected centre of mass to the left/right of the actual centre of mass of the lock. The Q factor of the pendulum was determined to be 170 ± 20 from finding the curve of best fit, or 160 ± 10, by counting the number of oscillations until the amplitude of its swing reduced to $e^{-\frac{\pi}{2}}$ of the release angle. These two values agree with each other as they are within each other's uncertainties.

The period of the pendulum was found to be dependent upon amplitude and string length, but independent of mass. The period's dependence upon amplitude only applies to large enough values of θ_0, since for $|\theta_0| \leq 0.5\,rad$, the value of C can be considered negligible (based on the uncertainties of the instruments used in this experiment). Generally, the relationship between the period and amplitude takes the form of a parabola opening upwards, centered at 0 rad. While there is a clear dependence of my pendulum's period on string length related by the following power law equation: $T = k(L_0 + L)^n$, the values of the fitted parameters (k = 1.83 ± 0.07, L_0 = -0.06 ± 0.01, n = 0.30 ± 0.04) and not consistent with the predicted values (k = 2, L_0 = 0, n = 0.5), and therefore my pendulum cannot be modelled by the predicted power law equation. Finally, there is no clear relationship between the period and mass of my pendulum. All collected data were consistent with the value of the calculated theoretical period. Plotting the residuals of the data also produced no clear trend/pattern. Therefore, the simplification that mass has no effect on period is a reasonable assumption to make for my pendulum.

Overall, the mathematical model of the pendulum's behaviour is supported by this experiment, given a fixed string length. The pattern displayed by the angular position data represents a sinusoidal curve that decays with time, and the behaviour of the pendulum's period displays independence from mass and amplitude at small enough values of θ_0, as predicted. It also displays dependence on string length, but the experimental results did not agree with the predicted behaviour; the given model for the relationship between period and string length cannot accurately model my pendulum.

APPENDIX A.

(1) Given $\tau \approx 78.4$ and $T \approx 1.49$ from the equation of the curve of best fit, the value of Q can be found:

$$Q = \pi \frac{\tau}{T}$$

$$Q = \pi \frac{78.4}{1.49}$$

$$\therefore Q \approx 170$$

(2) The value of the amplitude of the pendulum's motion can be modelled by the equation:

$A(t) = Ae^{-t/\tau}$,

where A is the amplitude and t is time.

By algebraically manipulating this equation, we can represent time t as:

$t = -\tau \ln\left(\frac{A(t)}{A}\right)$.

By substituting the equation for Q into the equation above and performing algebraic manipulation, we can find the number of oscillations, $\frac{t}{T}$, as a function of Q:

$$t = -\frac{QT\ln\left(\frac{A(t)}{A}\right)}{\pi}$$

$$\therefore \frac{t}{T} = -\frac{Q\ln\left(\frac{A(t)}{A}\right)}{\pi}$$

(3) In order to C to be considered insignificant, $C\theta_0^2 \leq 0.02, C = 0.059$

$$0.059\theta_0^2 \leq 0.02$$

$$\theta_0^2 \leq \frac{0.02}{0.059}$$

$$|\theta_0| \leq 0.582\, rad$$

Rounding up to keep one significant digit would give a value of 0.6 rad, but to stay conservative, it is better to round down to ensure that the value of C will have negligible effect on the period:

$$\therefore |\theta_0| \leq 0.5\, rad$$