VECTOR SPACE

- Composed of a set V and the two operations of vector addition and scalar multiplication
- Must satisfy the 8 axioms in Medici
- **Subspace of V** is a subset W \subseteq V that is itself a vector space
    - Has same operations of addition and multiplication as V
    - **Theorem**: a non-empty subset W is a subspace of V *iff* cx + y \in W when x, y \in W
- **5.3 Prop. 1**: if S is a subset of vector space V, then spanS is a subspace of V

DIMENSION

- Number of basis vectors for a vector space
    - **Fundamental Theorem – the number of vectors in a linearly independent list of vectors in V cannot exceed dimV**
- **Existence of Bases Theorem** (6.4 Theorem V) – any LI list of vectors can be extended to a basis for V
    - (6.4 Theorem VI) – if U is a subspace of n-dimensional V, then dimU <= n
        - dimU = n *iff* U = V
    - (6.4 Theorem VII) – any spanning set for V contains a basis for V
    - (6.4 Theorem VIII) – if dimV = n, then 1) any set of n vectors that is LI is a basis for V, and 2) any set of n vectors that spans V is a basis for V

RANK

- **Column Space** – span of columns of A {A**x** | **x** ER}
    - Can write matrix vector product Ax as LC of columns in matrix
    - Can write matrix product AB as a matrix where each column is a LC of columns in A
        - col AB \subseteq col A
        - **Lemma** – if each column in C is an LC of the columns of A, then there exists a matrix B such that C = AB
- **Row Space** – span of rows of A (= span of columns of A^T = {A^T**x** | **x** ER})
    - Row operations change column space, don't change row space
- **Rank** – rank A = dim col A (number of LI columns of A)
- **Theorem** – A is invertible *iff* col A = nR
- **Theorem** – A is invertible *iff* rank A = n
- **Rank Theorem** – for any matrix A: dim col A = dim row A = rank A

- (7.4 Property II) – rank A = rank A^T
  - **Theorem** – rank A = rank A^T = rank A^TA = rank AA^T
- (7.4 Property III) – rank AB <= min{rank A, rank B}
- **Theorem** – rank A <= rank [A|b], and Ax = b has a solution *iff* rank A = rank [A|b]
- **Full Row/Column Rank** (7.4 Theorem IV and V) – rank A = number of rows/columns
  - All rows/columns are LI
- **7.2 Prop 1 Theorem** – if B has full column rank, then rank A = rank BA
  - Multiplying a matrix on the left by a matrix with full column rank doesn't change its rank
- **Theorem** – if C has full row rank, then rank A = rank AC
  - Multiplying a matrix on the right by a matrix with full row rank doesn't change its rank
- **Theorem** – A has full column rank *iff* A^TA is invertible, A has full row rank *iff* AA^T is invertible

NULL SPACE

- **Null Space** – set of solutions to the homogeneous system Ax = 0
  - {x | Ax = 0}
- **Nullity** – nullity A = dim null A
- **Rank-Nullity Theorem** (7.4 Theorem II) – nullity A = n – rank A

COORDINATES

- **Theorem** – $[cx + y]_a = c[x]_a + [y]_a$
- **8.4 Theorem I** – coordinates preserve linear independence
- **Change of Basis Matrix** – $[x]_b = P_{ba}[x]_a$ where columns of P_{ba} are coordinate vectors of basis vectors for *a* WRT basis *b*
  - **Theorem** – Inverse of matrix is P_{ab}

DETERMINANT

- **A** determinant function – unique, alternating, multilinear
  - *Any* alternating multilinear function *f* satisfies f(A) = (detA)*f*(I)

- **THE Determinant** – unique, alternating, multilinear function on the rows of matrix A, where its value on the identity matrix I is 1
    - Determinant of a 2x2 matrix det[a b ; c d] = ad – bc
    - If A has equal rows, then detA = 0
    - If A has an entire row of zeros, then detA = 0
    - If rows of A are LD, then detA = 0
- **ij minor** of an nxn matrix A – omit the i^{th} row and j^{th} column of A ($\mathbf{A}_{ij}$)
- **Theorem** - $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$ - expanding along the rows
    - Can also expand along columns
- *Theorem VI.* INVERTIBILITY THEOREM: *Let* $\mathbf{A} \in {}^n\mathbb{R}^n$. *Then* $\mathbf{A}$ *is invertible if and only if* $\det \mathbf{A} \neq 0$.
    - If A is not full rank, det A = 0
    - If detA != 0, rows of A are LI
- **Product Theorem** – if A and B be nxn matrices, then detAB = detA detB
- **Theorem** – if A is invertible then det $A^{-1} = (\det A)^{-1}$
- **Adjoint of A** – nxn matrix whose ij entry is $(-1)^{i+j} \det A_{ji}$
- **Theorem** – A(adj A) = (det A)I
    - if A is invertible, $A^{-1} = (1/\det A)\text{adj}A$
- **Transpose Theorem** – det A^T = det A
- **Cramer's Rule** – solution to Ax = b is: $x_i = \frac{\det B_i}{\det A}$
    - B_i is same as A but replace ith column of A by b

EIGENVALUES AND EIGENVECTORS

- **Eigenvector of A (nxn matrix)**: a NON-ZERO vector where $Ax = \lambda x$
    - λ is an **eigenvalue** of A corresponding to x
    - **Eigenspace corresponding to eigenvalue $\boldsymbol{\lambda}$**: subspace containing set of all eigenvectors corresponding to that eigenvalue (together with the zero vector)
    - $\boldsymbol{E_\lambda(A)} = \{x \in {}^nR \mid Ax = \lambda x\} = \boldsymbol{null(\lambda I - A)}$
- **10.2 Prop. 1**: the intersection of two different eigenspaces is the zero vector
- **Characteristic polynomial of A**: $p_A(\lambda) = \det(\lambda I - A)$
    - $c_{\mathbf{A}}(\lambda) = \lambda^n - (\text{tr}\,\mathbf{A})\lambda^{n-1} + \cdots + (-1)^n \det \mathbf{A}$
    - Degree of exactly n

- As a result: an nxn matrix A can have at most n distinct eigenvalues
- **TO SOLVE FOR EIGENVALUES AND EIGENVECTORS:**
  - λ is an eigenvalue *iff* $\lambda I - A$ is non-invertible *iff* $\det(\lambda I - A) = 0$ *iff* $p_A(\lambda) = 0$
  - *iff* λ is a ROOT of characteristic polynomial
  - For eigenvector/eigenspaces: $E_\lambda(A) = null(\lambda I - A)$

DIAGONALIZATION

- A and B (nxn) are similar if there exists an invertible nxn matrix S such that A = SBS^{-1}
  - Same determinant, rank, characteristic polynomial, eigenvalues, and trace
- **Diagonal Matrix**: all entries except for diagonal are 0
- **Diagonalizable Matrix**: nxn matrix A similar to an nxn diagonal matrix D (A = SDS^{-1})
  - Diagonal entries of D are eigenvalues
  - S is change of basis matrix from eigenbasis to standard basis
  - **Diagonalization Theorem**: A is diagonalizable *iff* A's eigenvectors are a basis for ^{n}R
- **10.3 Theorem II**: if A is an nxn matrix with exactly n distinct eigenvalues, then detA is the product of the eigenvalues and the trA is the sum of the eigenvalues
- **MULTIPLICITIES**:
  - **Algebraic of $\boldsymbol{\lambda}$:** number of times λ appears as a root of $p_A(\lambda)$
  - **Geometric of $\boldsymbol{\lambda}$:** $\dim E_\lambda(A)$
  - **Multiplicity Theorem:** 1 <= geometric <= algebraic
- **CONDITIONS FOR DIAGONALIZATION:**
  - Diagonalization Theorem
  - **10.4 Theorem III**: if A (nxn) has n distinct eigenvalues, then A is diagonalizable
  - **10.5 Theorem VI (Diagonalization Test)**: A (nxn) is diagonalizable *iff* the geometric and algebraic multiplicities of each eigenvalue are equal

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Theorem: Let A be a $n \times n$ diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct). Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for $\mathbb{R}^n$ consisting of eigenvectors of A. If $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_n\mathbf{v}_n$, then the system $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ has solution

$$\mathbf{x}(t) = e^{\lambda_1 t}(c_1\mathbf{v}_1) + e^{\lambda_2 t}(c_2\mathbf{v}_2) + \cdots + e^{\lambda_n t}(c_n\mathbf{v}_n)$$