ESC103 SUMMARY

Overview of the Course

4 central problems of linear algebra:

1. $A\vec{x} = \vec{b}$: Linear systems, solve for $\vec{x}$
2. $A\vec{x} = \vec{b}$: Least squares, finding an approximation for $\vec{x}$
3. $A\vec{x} = \lambda\vec{b}$: Finding eigenvalues and eigenvectors
4. $A\vec{x} = \sigma\vec{b}$: Singular value problems (did not cover)

Matrices

$$A + B = B + A$$

$$c(A + B) = cA + cB$$

$$A + (B + C) = (A + B) + C$$

$$C(A + B) = CA + CB$$

$$(A + B)C = AC + BC$$

$$A(BC) = (AB)C$$

Generally: $AB \neq BA$

If A is square: $A^p A^q = A^{p+q}$

Linear Transformations

A linear transformation, L, maps a vector in R^n to R^m with the following properties:

1. $L(c\vec{v}) = cL(\vec{v})$
2. $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$

All linear transformations can be summarized by matrices. A transformation applied to a vector is represented by the matrix multiplication of that vector.

Each column of the matrix can be determined by applying the transformation to each of the basis vectors – where the basis vectors land after the transformation. The transformed vector will be a linear combination of the transformed basis vectors.

Some important transformations/matrices:

1. Identity matrix, I
   a. $I(\vec{w}) = \vec{w}$
   b. $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
2. Zero matrix, O
   a. $O(\vec{w}) = \vec{0}$
   b. $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

<u>Composition of Linear Transformations</u>

Multiple linear transformations can be summarized by a single matrix, where the matrix is the product of the matrices that describe the individual transformations.

<u>Matrix Inverses</u>

The inverse of a matrix 'undoes' the transformation done by the matrix:

$$M^{-1}M = I$$

If the transformation described by matrix M squishes everything into a lower dimension, the inverse does not exist – if several vectors transform to a single vector, you cannot transform a single vector back to several vectors (similar to how functions and inverse functions must be one-to-one).

Properties of matrix inverses:

1. If A^{-1} exists, then using GE to find the solution to $A\vec{x} = \vec{b}$ produces the RNF.
2. If there is a non-zero vector such that $A\vec{x} = \vec{0}$, then A cannot have an inverse.
3. If A and B are both invertible and the same size, then $(AB)^{-1} = B^{-1}A^{-1}$.
   a. **Remember: order is reversed**
4. If A is invertible, then so is A^{-1}, and $(A^{-1})^{-1} = A$
5. Every invertible matrix can be expressed as a product of elementary matrices.

The Determinant (of a linear transformation/matrix)

The scalar factor by which a linear transformation scales an area/volume.

For a 2x2 matrix, $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\det(M) = ad - bc$$

If $\det(M) = 0$, the linear transformation described by M squishes everything into a smaller dimension (e.g. a linear transformation described by projection onto a line squishes everything into 1D).

Therefore, if $\det(M) = 0$, then matrix M has no inverse.

Eigenvectors and Eigenvalues (of a linear transformation)

$$M\vec{w} = \lambda\vec{w}$$

Eigenvector – a vector that changes by a scalar factor, λ (the eigenvalue), when transformed. A vector that **does not change direction** when transformed.

To solve for the eigenvalues (and then eigenvectors) of a linear transformation:

$$\det(M - \lambda I) = 0$$

Gaussian Elimination

3 possible outcomes:

1. Unique Solution – every variable in RNF is a leading variable
2. Infinite Solutions – at least 1 free variable
3. No Solutions – occurs when a row a zeros followed by a nonzero number appears

Classifying systems of equations:

- ‘Consistent’ = Has solutions
- ‘Inconsistent’ = No solutions

Rank

For an $m \times n$ matrix A, 'full rank' means: $rank(A) = \min(m, n)$

4 possibilities for linear systems ($A\vec{x} = \vec{b}$):

Full Rank	Square	$m = n$	1 solution
	Short and Wide	$m < n$	∞ solutions
	Tall and Thin	$m > n$	0 or 1 solutions
Not Full Rank			0 or ∞ solutions

Transpose

Properties:

1. $(A^T)^T = A$
2. $(cA)^T = cA^T$
3. $(A + B)^T = A^T + B^T$
4. $(AB)^T = A^T B^T$
5. If matrix A is invertible (square), so is A^T, and $(A^T)^{-1} = (A^{-1})^T$

Least Squares Problem

When $A\vec{x} = \vec{b}$ has no solution – when $\vec{b}$ is not within the column space of A, meaning that there is no linear combination of the columns of A that can produce $\vec{b}$, and therefore $A\vec{x} \neq \vec{b}$.

If $\vec{b}$ lies within the column space of A, there is a solution. If it sticks out of the column space of A, we want to find $\overrightarrow{x_{LS}}$ such that $A\overrightarrow{x_{LS}} \cong \vec{b}$. The closest we can get to $\vec{b}$ is the projection of $\vec{b}$ onto the column space of A.

We define the error vector (between $\vec{b}$ and its approximation): $\vec{e} = \vec{b} - A\overrightarrow{x_{LS}}$. This vector sticks orthogonally out of the column space of A – it is orthogonal to every column vector of A.

Therefore:

$$A^T \vec{e} = \vec{0}$$

$$A^T \left(\vec{b} - A\overrightarrow{x_{LS}}\right) = \vec{0}$$

Rearranging gives us the 'Normal Equations':

$$A^T A\overrightarrow{x_{LS}} = A^T b$$

$$\therefore \overrightarrow{x_{LS}} = (A^T A)^{-1} A^T \vec{b}$$

Numerical Solutions to Differential Equations

Euler's Method:

$$Y_{n+1} = Y_n + \Delta t Y_n'$$

Improved Euler's Method:

$$Y_{n+1} = Y_n + \Delta t S$$

$$S = \frac{Y_n' + Y_{n+1}'}{2}$$

$$Y_{n+1} = Y_n' + \Delta t Y_n'$$

A is the 'State Matrix': $Y' = AY$

Finding the Inverse Using Gaussian Elimination

Each elementary row operation performed in GE can be represented by an elementary matrix. An elementary matrix can be found by applying the row operation it summarizes to I:

1. Interchange 2 rows
   a. Example: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
2. Multiply one row by a non-zero constant:
   a. Example: multiplying top row by c $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$
3. Add/subtract a multiple of one row to/from another row:
   a. Example: adding c*row2 to row1: $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$

Algorithm:

$$[A|I] \xrightarrow{GE} [I|A^{-1}]$$

A^{-1} can be represented as a product of elementary matrices.

A can be represented as a product of the inverses of the elementary matrices.

LU Decomposition

Procedure for solving $A\vec{x} = \vec{b}$:

1. Rewrite $A\vec{x} = \vec{b}$ as $LU\vec{x} = \vec{b}$
2. Rewrite original system as $L\vec{y} = \vec{b}$ (where $\vec{y} = U\vec{x}$), then solve for $\vec{y}$
3. We now know $\vec{y}$, so $U\vec{x} = \vec{y}$ and solve for $\vec{x}$

Factorization Phase of LU Decomposition:

- We can bring matrix A to U (end of forward step of GE) by applying k elementary row operations to A:

$$E_k E_{k-1} \cdots E_2 E_1 A = U$$

$$A = E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1} U$$

$$\therefore L = E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1}$$

- Procedure:
  - Perform the forward step of GE on A
  - Keep track of multipliers
  - Every time an elementary operation is performed on an element of A, place the reciprocal or opposite of that operation in L in the same spot as the element being operated on