ECE259 - Electromagnetism

Charges and shit

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Contents

PART

I

Electrostatics

In electrostatics, charges are at rest and electric fields are constant over time. There are three ways of finding an electric field $\vec{E}$:

1. Coulomb's Law
2. Gauss' Law
3. Potential

Section 1

Introduction

Subsection 1.1

Charge

- a fundamental property of matter – due to excess (-ve charge) or deficit (+ve) of electrons [1]
- charge is conserved (cannot be created/destroyed)
- symbol: q or Q

[1] *Charge of an electron:* $e = -1.6 \cdot 10^{-19} C$

Subsection 1.2

Current

Definition 1

Current: rate of charge flow across a finite area.

$$I = \frac{dq}{dt} \tag{1.1}$$

- units: $\left[\frac{C}{S}\right] = [A]$ (Amperes)

Section 2

Coulomb's Law

Theorem 1

Coulomb's Law: The force between two point charges is equal to:

$$\overrightarrow{\mathbf{F_{12}}} = k\frac{q_1 q_2}{R_{12}^2}\overrightarrow{\mathbf{a_{R_{12}}}} \tag{2.1}$$

where $\overrightarrow{\mathbf{F_{12}}}$ is the force exerted by q_1 on q_2, and

$$k = \frac{1}{4\pi\epsilon_0} = \frac{1}{4\pi(8.85 \cdot 10^{-12})} \approx 9 \cdot 10^9 Nm^2/C^2 \tag{2.2}$$

- ϵ_0 is the permittivity of vacuum

Definition 2 **Electric Field Intensity**: field of force per unit +ve charge (units: N/C).

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q} \quad (2.3)$$

$$\implies \vec{\mathbf{F}} = q\vec{\mathbf{E}} \quad (2.4)$$

- Direction of electric field is the same as the force direction exprienced by the test charge
- Electric field is independent of the test charge
- Field can exist in vacuum

2.0.1 Electric field due to a point charge

- if the point source charge is at the origin:

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q_{test}} = \frac{1}{4\pi\epsilon_0}\frac{q_{src}\cancel{q_{test}}}{R^2\cancel{q_{test}}}\overrightarrow{\mathbf{a_R}} \quad (2.5)$$

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{q_{src}}{R^2}\overrightarrow{\mathbf{a_R}} \quad (2.6)$$

- if the point source charge is NOT at the origin: [2]

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{\mathbf{R}} - \vec{\mathbf{R'}}|^2}\overrightarrow{\mathbf{a_{qp}}} \quad (2.7)$$

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{\mathbf{R}} - \vec{\mathbf{R'}}|^3}(\vec{\mathbf{R}} - \vec{\mathbf{R'}}) \quad (2.8)$$

[2] *we will use prime notation to represent a source*

2.0.2 Electric field due to system of discrete charges

- use vector superposition:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\sum_k \frac{q_k}{|\vec{\mathbf{R}} - \overrightarrow{\mathbf{R'_k}}|^3}(\vec{\mathbf{R}} - \overrightarrow{\mathbf{R'_k}}) \quad (2.9)$$

2.0.3 Electric field due to continuous distribution of charge

- recall: the electric field due to discrete charges is simply equal to the sum of the effects from each charge

- in a continuous distribution, we take the integral over the distribution, summing up the contribution from each tiny element of charge:

$$\vec{\mathbf{E}} = \int_{distribution} d\vec{\mathbf{E}} \quad (2.10)$$

- a differential amount of charge can be represented as $dQ = \rho_V dV'$ in a volume, $dQ = \rho_S dS'$ on a surface, or $dQ = \rho_l dl'$ along a line

- For a **volume charge**:

$$\vec{\mathbf{E}} = \int_{V'} d\vec{\mathbf{E}} = \int_{V'} \frac{1}{4\pi\epsilon_0} \frac{\rho_V dV'}{|\vec{\mathbf{R}} - \vec{\mathbf{R'}}|^2} \overrightarrow{\mathbf{a}_{\mathbf{R}-\mathbf{R}'}} \tag{2.11}$$

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho_V}{|\vec{\mathbf{R}} - \vec{\mathbf{R'}}|^3} (\vec{\mathbf{R}} - \vec{\mathbf{R'}}) dV' \tag{2.12}$$

- Similarly, for a **surface charge**:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \int_{S'} \frac{\rho_S}{|\vec{\mathbf{R}} - \vec{\mathbf{R'}}|^3} (\vec{\mathbf{R}} - \vec{\mathbf{R'}}) dS' \tag{2.13}$$

- and for a **line charge**:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \int_{l'} \frac{\rho_S}{|\vec{\mathbf{R}} - \vec{\mathbf{R'}}|^3} (\vec{\mathbf{R}} - \vec{\mathbf{R'}}) dl' \tag{2.14}$$

2.0.4 STEPS FOR SOLVING CHARGE DISTRIBUTION PROBLEMS

1. Choose appropriate coordinate system (depends on symmetry of charge distribution)
2. Find expression for differential charge element, dQ
3. Find expression for $\vec{\mathbf{R}} - \vec{\mathbf{R'}}$
4. Write out the integral expression for $d\vec{\mathbf{E}}$
5. Integrate the expression (pay attention to changing unit vectors in cylindrical and spherical coordinates)

SECTION 3

Gauss's Law

SUBSECTION 3.1

Integral form

Some quick notes on field lines and flux:

- direction of an electric field is tangential to the field lines
- magnitude of an electric field is proportional to the line density
- the electric flux through a surface:

$$\Phi = \int_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} \tag{3.1}$$

Theorem 2 **Gauss's Law (integral form)**: the total electric flux out of a surface is equal to the total charge enclosed by the surface divided by the permittivity of vacuum.

$$\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{Q_{enclosed}}{\epsilon_0} \tag{3.2}$$

- $\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} > 0 \implies$ net flux out $\implies$ +ve charge enclosed
- $\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} < 0 \implies$ net flux in $\implies$ -ve charge enclosed
- units for flux: [Vm]

- to solve for the electric field intensity, evaluate the integral in Gauss's Law and isolate for $\vec{\mathbf{E}}$

- Electric field due to a +ve:
  - point charge ($\propto \frac{1}{R^2}$):

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\epsilon_0 R^2}\overrightarrow{\mathbf{a_R}} \tag{3.3}$$

  - infinitely long line charge ($\propto \frac{1}{R}$):

$$\vec{\mathbf{E}} = \frac{\rho_l}{2\pi\epsilon_0 R}\overrightarrow{\mathbf{a_R}} \tag{3.4}$$

  - infinite plane charge (constant):

$$\vec{\mathbf{E}} = \frac{\rho_S}{2\epsilon_0} \tag{3.5}$$

SUBSECTION 3.2

Differential form

Some quick notes on divergence:

Definition 3

Divergence: the divergence of a vector field $\vec{\mathbf{A}}$ can be thought of as its net outward flux per unit volume as volume approaches 0, i.e. its net outward flux at a point.

$$\vec{\nabla} \cdot \vec{\mathbf{A}} \equiv \lim_{\Delta V \to 0} \frac{\oint_S \vec{\mathbf{A}} \cdot d\vec{\mathbf{S}}}{\Delta V} \tag{3.6}$$

Theorem 3

Divergence Theorem: integrating the divergence over a volume gives the net outward flux over the surface area enclosing the volume.

$$\int_V \vec{\nabla} \cdot \vec{\mathbf{A}} dV = \oint_S \vec{\mathbf{A}} \cdot d\vec{\mathbf{S}} \tag{3.7}$$

- Substituting in electric field $\vec{\mathbf{E}}$ for $\vec{\mathbf{A}}$, we obtain

$$\int_V \vec{\nabla} \cdot \vec{\mathbf{E}} dV = \oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} \tag{3.8}$$

- but we notice that the right side appears in the integral form of Gauss's Law, so

$$\int_V \vec{\nabla} \cdot \vec{\mathbf{E}} dV = \frac{Q}{\epsilon_0}. \tag{3.9}$$

- Noticing that we can write Q as a volume integral of charge density, we get:

$$\int_V \vec{\nabla} \cdot \vec{\mathbf{E}} dV = \frac{\int_V \rho_V dV}{\epsilon_0} \tag{3.10}$$

- and therefore...

Theorem 4 **Gauss's Law (differential form)**:

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho_V}{\epsilon_0} \tag{3.11}$$

- left side can be thought of as the net outward electric flux at a point

A summary of Gauss's Law:

Theorem 5 **Gauss's Law**: for a given volume and its enclosing surface, Gauss's Law relates the enclosed charge to the electric field it produces.

$$\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{Q_{enclosed}}{\epsilon_0} \tag{3.12}$$

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho_V}{\epsilon_0} \tag{3.13}$$

Section 4

Electric Potential

Definition 4 **Electric Potential**: The amount of work needed to move a unit of electric charge from a reference point to a specific point in an electric field.

$$\Delta V_{AB} = V_B - V_A = \frac{\Delta U_{AB}}{q} = -\int_A^B \vec{E} \cdot d\vec{l}.$$

Some notes on potential:

1. Potential is relative (t a reference potential where $V_{ref} = 0$)
2. Potential is associated with the field, not the test charge (it is independent of the test charge)
3. Units in Volts (V)
4. Analogy: charge going against an $\vec{E}$ field is like a person moving up a hill
   - Potential can be thought of as altitude – equipotential lines can be thought of as lines of equal altitude
   - Electric field lines are perpendicular to equipotential lines – they are the fastest way to change altitude
5. Differential form:

Definition 5 **Differential form of Electric Potential**:

$$\vec{E} = -\vec{\nabla} V.$$

- proves that electric fields are conservative

- Electric potential (reference taken at ∞) for **discrete cases**:

$$V(R) = \frac{1}{4\pi\epsilon_0} \sum_k \frac{q_k}{|\vec{R} - \vec{R}'_k|}.$$

- Electric potential (reference taken at ∞) for **continuous cases**:

$$V(R) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho_{V'}}{R} dV'$$
$$V(R) = \frac{1}{4\pi\epsilon_0} \int_{S'} \frac{\rho_{S'}}{R} dS'$$
$$V(R) = \frac{1}{4\pi\epsilon_0} \int_{l'} \frac{\rho_{l'}}{R} dl'.$$

  - where R is the distance from the source charge to a point of interest

Section 5

Perfect Conductors

- conductors have free charges
- Inside a perfect conductor:

$$\rho_V = 0$$
$$\vec{E} = 0$$
$$\implies V = \text{constant}.$$

- Boundary conditions at a perfect conductor/free space interface:

$$E_t = 0$$
$$E_n = \frac{\rho_s}{\epsilon_0}.$$

  - electric field is always perpendicular to the boundary of a perfect conductor

Section 6

Dielectrics

- in dielectrics, charges are bound – dielectrics can be polarized
- applying a static electric field to a dielectric material can induce a dipole, creating a dipole moments
  - the **dipole moment** of two equal and opposite charges is defined as $\vec{p} = q\vec{d}$, where q is the magnitude of the charges and $\vec{d}$ is the distance between the two charges
- some materials are made of molecules that have non-zero dipole moments (e.g. water)
- some materials can exhibit permanent electric dipole moment in the absence of an external electric field – called "electrics"

Subsection 6.1

Polarization Vector

Definition 6 **Polarization Vector**: the volume density of electric dipole moment

$$\vec{P} = \lim_{\Delta V \to 0} \frac{\sum_{k=1}^{n\Delta V} \vec{p_k}}{\Delta V}.$$

- n is number of particles per unit volume so $n\Delta V$ is total number of particles
- units: $[\mathrm{C/m^2}]$

- the charge density of a dielectric is given by the following formulas:

Theorem 6 **Charge density of a dielectric** (referred to as **polarization charge densities** or **bound charge densities**):

- for surface charge density:

$$\rho_{PS} = \vec{P} \cdot \vec{a_n}.$$

- for volume charge density:

$$\rho_{PV} = -\vec{\nabla} \cdot \vec{P}.$$

- a polarized dielectric can be replaced by an equivalent polarization surface charge density ρ_{PS} and polarization volume charge density ρ_{PV} for calculations:

$$V = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\rho_{PS}}{|\vec{R} - \vec{R}'|} dS' + \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho_{PV}}{|\vec{R} - \vec{R}'|} dV'.$$

Section 7

Electric Flux Density and Dielectric Constant

Definition 7 **Electric Flux Density** (or electric displacement):

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}.$$

Theorem 7 **Generalized Gauss' Law**:

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\oint_S \vec{D} \cdot d\vec{S} = Q_{encl}.$$

- for linear isotropic materials (where $\vec{\boldsymbol{P}}$ and $\vec{\boldsymbol{E}}$ are proportional – point in same direction):

$$\vec{\boldsymbol{P}} = \epsilon_0 \chi_e \vec{\boldsymbol{E}}.$$

  - where $\chi_e = \epsilon_r - 1$ is the electrical susceptibility (unitless)

- then we get

$$\vec{\boldsymbol{D}} = \epsilon_0(1+\chi_e)\vec{\boldsymbol{E}} = \epsilon_0\epsilon_r\vec{\boldsymbol{E}} = \epsilon\vec{\boldsymbol{E}}.$$

  - where $\epsilon_r = 1+\chi_e = \frac{\epsilon}{\epsilon_0}$ is the **relative permittivity** or **dielectric constant** of the medium (dimensionless quantity)
  - where ϵ is the **absolute permittivity** of the medium

SECTION 8

Boundary Conditions for Electrostatic Fields

- we already know the boundary conditions between conductor/free space interfaces
- we now determine the boundary conditions at the interface between two generic dielectric media:

$$E_{t1} = E_{t2}$$
$$(\vec{\boldsymbol{D_1}} - \vec{\boldsymbol{D_2}}) \cdot \vec{\boldsymbol{a_{n2}}} = \rho_S.$$

SECTION 9

Capacitors

- **capacitor**: device consisting of two isolated conductors for storing electrostatic potential energy
- a charged capacitor has equal but opposite charge on the two conductors
  - the charge of a capacitor refers to the charge on one conductor
- the amount of energy stored is the energy it takes to charge a capacitor from a discharged state

SUBSECTION 9.1

Capacitance

Definition 8 **Capacitance**:

$$C = \frac{Q}{V}.$$

- Q is charge of the capacitor
- V is the difference in potential between the two conductors
- units: [C/V] = [F] (Farads)

- Capacitance is independent of Q and V – it's only dependent on the physical attributes of the capacitor
  - dimension, shape, dielectric material

9.1.1 Calculating Capacitance

1. Choose coordinate system
2. Assume $+Q/-Q$ on conductors
3. Find electric field from Q distribution
4. Find $V_{AB} = -\int_A^B \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}$ where A carries $-Q$ and B carries $+Q$
5. $C = \frac{Q}{V}$

- for a parallel plate capacitor:

$$C = \frac{S\epsilon}{d}.$$

  - where S is plate area, d is distance between plates, ϵ is dielectric permittivity

Subsection 9.2

Inhomogeneous Capacitors (Series and Parallel Connections)

- capacitors in **series**:

$$C = \left(\frac{1}{C_1} + \frac{1}{C_2} + \dots\right)^{-1}$$
$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \dots.$$

- capacitors in **parallel**:

$$C = C_1 + C_2 + \dots.$$

PART II

Solution of Electrostatic Problems

- previously we've been given the charge distribution everywhere to find $\vec{D}$, $\vec{E}$, V, etc.
- in more practical problems, we do not know the charge distribution everywhere

Section 10

Poisson's Equation

Theorem 8 **Poisson's Equation:**

$$\nabla^2 V = -\frac{\rho}{\epsilon}.$$

- in the case that $\rho = 0$, this equation becomes **Laplace's Equation:**

$$\nabla^2 V = 0.$$

Section 11

Uniqueness of Electrostatic Solutions

Theorem 9 **Uniqueness Theorem**: there is only one solution to Poisson's equation (and Laplace's equation) for a given set of sources and boundary conditions.

Section 12

Method of Images

- method of images is a technique for solving electrostatics problems in the presence of perfect conductors without solving Poisson's or Laplace's equations
- **image theory** states that a given charge configuration above an infinite grounded perfect conducting plane may be replaced by the charge configuration itself, its image, and an equipotential surface in place of the conducting plane
- **image method:** 2 conditions must always be satisfied:
  1. image charges must be located in the conducting region
  2. image charges must be located such that on the conducting surface the potential is zero or constant
     - this is equivalent to the boundary condition that says the tangential component of the electric field vanishes on the surface of a PEC

PART III

Steady Electric Currents

Section 13

Current Density and Ohm's Law

- conduction current: in conductors and semiconductors due to motion of electrons and holes
- **average drift velocity** is defined as:

  $$\vec{u} = -u_e \vec{E}.$$

  where u_e is electron mobility $[m^2/sV]$
- **current**: amount of charge through S per unit time

Definition 9 **Current Density**: a vector whose magnitude is the electric current per cross-sectional area at a given point in space, direction is the motion of positive charges at that point (direction is same as $\vec{u}_e$)

$$\vec{J} = \rho_e \vec{u}_e = -\rho_e u_e \vec{E}.$$

- notice that $\vec{J}$ is proportional to $\vec{E}$
- integrating $\vec{J}$ over an area give s you the current flowing through that area:

$$I = \int_S \vec{J} \cdot d\vec{S}.$$

Theorem 10 **Ohm's Law**:

$$\vec{J} = \sigma \vec{E}.$$

- σ is known as the conductivity

Section 14

Power Dissipation and Joule's Law

Theorem 11 **Joule's Law**: the total power dissipated over a volume V is

$$P = \int_V \vec{E} \cdot \vec{J} dV.$$

- notice that

$$P = \int_V \vec{E} \cdot \vec{J} dV = \int_l \vec{E} \cdot d\vec{l} \int_S \vec{J} \cdot d\vec{S} = VI.$$

Subsection 14.1

Resistance

- recall $\vec{J} = \sigma \vec{E}$

Definition 10 **Resistance**:

$$R \equiv \frac{V}{I} = \frac{\int_l \vec{E} \cdot d\vec{l}}{\int_S \vec{J} \cdot d\vec{S}} = \frac{\int_l \vec{E} \cdot d\vec{l}}{\int_S \sigma \vec{E} \cdot d\vec{S}}.$$

- regardless of how resistance is defined, it is **independent** of V and I
- R is dependent on the physical attributes of the resistor

Steps to calculate resistance

1. choose coordinate system
2. assume V_0 = potential drop between terminals
3. Find $\vec{E}$ from V
4. Find $I = \int_S \vec{J} \cdot d\vec{S} = \int_S \sigma \vec{E} \cdot d\vec{S}$
5. $R = V_0/I$

Section 15

Continuity Equation (Kirchoff's Current Law)

Theorem 12 **Continuity Equation**: the divergence of the current density is equal to the change in charge density:

$$\vec{\nabla} \cdot \vec{J} = -\frac{d\rho}{dt}.$$

- at steady state:

$$\frac{d\rho}{dt} = 0 \quad \Longrightarrow \vec{\nabla} \cdot \vec{J} = 0 \quad \Longrightarrow \oint_S \vec{J} \cdot d\vec{S} = 0.$$

- this gives us **Kirchoff's current law**:

$$\sum_j I_j = 0 \quad \text{at steady state.}$$

Section 16

Boundary Conditions for Current Density

- at steady state, we have that

$$\vec{\nabla} \cdot \vec{J} = 0 \quad \text{and} \quad \vec{\nabla} \times \left(\frac{\vec{J}}{\sigma}\right) = 0.$$

  - second one is because curl of $\vec{E}$ is 0
- this gives us the following **boundary conditions for current density**:
  - normal component of $\vec{J}$ is continuous across the boundary (from $\vec{\nabla} \cdot \vec{J} = 0$):

$$J_{1n} = J_{2n}.$$

  - tangential components have the same ratio as the conductivities of the two materials:

$$\frac{J_{2t}}{J_{1t}} = \frac{\sigma_2}{\sigma_1}.$$

PART IV

Static Magnetic Fields

Section 17

Introduction

- the force exerted on a moving charge q by a magnetic field $\vec{B}$ is

$$\vec{F}_m = q\vec{u} \times \vec{B}.$$

- so the total electromagnetic force on a charge is given by

$$\vec{F} = q\vec{E} + q\vec{u} \times \vec{B}.$$

Section 18

Fundamental Postulates of Magnetostatics in Free Space

Theorem 13 **Fundamental Postulates of Magnetostatics in free space**:

- in differential form:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{(no monopoles)}$$
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \text{(Amperes Law).}$$

  where μ_0 is the permeability of free space (constant)

- in integral form:

$$\oint_S \vec{B} \cdot d\vec{S} = 0$$
$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I.$$

- the first postulate tells us that there are no magnetic flow sources (unlike electric fields) – all magnetic flux lines always close upon themselves
  - also known as the law of conservation of magnetic flux because it states taht the total outward magnetic flux through any closed surface is 0
- second postulate is a form of Ampere's Law and tells us that the circulation of the magnetic flux density in free space around any closed path is proportional to the total current flowing through the surface bounded by the path

Section 19

Magnetic Vector Potential

- $\vec{B}$ can be expressed as the curl of another vector field, say $\vec{A}$:

Definition 11 **Magnetic Vector Potential** $\vec{A}$ is defined such that

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

- since $\vec{A}$ is just a mathematical construct, we're free to choose

$$\vec{\nabla} \cdot \vec{A} = 0.$$

- then we have the **Vector Poisson's equation**:

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}.$$

- solving this equation, we obtain an alternate form for $\vec{A}$:

$$\vec{A}(\vec{R}) = \frac{\mu_0}{4\pi} \int_{v'} \frac{\vec{J}(\vec{R}')}{|\vec{R} - \vec{R}'|} dv'.$$

Section 20

Biot-Savart Law

- in many applications we want to determine the magnetic field due to a current-carrying circuit (currents confined in wires)
- for a thin wire, we have that

$$\vec{J} dv' = \vec{J} S dl' = I d\vec{l'}.$$

- and so the previous expression we had for $\vec{A}$ becomes

$$\vec{A}(\vec{R}) = \frac{\mu_0 I}{4\pi} \int_{C'} \frac{d\vec{l'}}{|\vec{R} - \vec{R}'|}.$$

- plugging this into $\vec{B} = \vec{\nabla} \times \vec{A}$ gives us the Biot-Savart Law

Theorem 14 **Biot-Savart Law**: formula for determining $\vec{B}$ caused by a current I in a closed path C'

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_{C'} \frac{d\vec{l'} \times (\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3}.$$

- can think of it as

$$\vec{B} = \int_{C'} d\vec{B}.$$

where $d\vec{B}$ is the contribution to $\vec{B}$ from current element $I d\vec{l'}$, given by:

$$d\vec{B} = \frac{\mu_0 I d\vec{l'}}{4\pi} \times \frac{(\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3}.$$

- a magnetic dipole is a small current loop
- for a magnetic dipole of radius b carrying current I:

$$\vec{B} = \frac{\mu_0 I b^2}{4R^3}(2\cos\theta \vec{a}_R + \sin\theta \vec{a}_\theta).$$

  - notice the similarity to the electric dipole

Section 21

Magnetization and Equivalent Current Densities

- recall that a magnetic dipole is a small current loop

Definition 12 **Magnetic Dipole Moment**:

$$\vec{m} = I\pi b^2 \vec{a}_z.$$

- direction is defined by current direction and the right hand rule

- we can think of atoms as microscopic magnetic dipoles (orbiting electrons and electron spin produce magnetic dipole moment)
- if we apply an external magnetic fields, it aligns all microscopic dipoles (since it exerts a torque)
- this is called **magnetization**

Definition 13 **Magnetization Vector**: volume density of magnetic dipole moment

$$\vec{M} = \lim_{\Delta V \to 0} \frac{\sum_{k=1}^{n\Delta V} \vec{m}_k}{\Delta V}.$$

- we can write the magnetic vector potential $\vec{A}$ as

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{v'} \frac{\vec{\nabla}' \times \vec{M}}{|\vec{R} - \vec{R}'|} dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\vec{M} \times \vec{a'}_n}{|\vec{R} - \vec{R}'|} dS'.$$

  - the first term is the contribution to $\vec{A}$ from a volume current density
  - second term is contribution from surface current density

- then the **equivalent magnetization current densities** are given by:

$$\vec{J}_m = \vec{\nabla}' \times \vec{M}$$
$$\vec{J}_{ms} = \vec{M} \times \vec{a}_n.$$

Section 22

Magnetic Field Intensity and Relative Permeability

Section 23

Amperes Law